G-valued pseudocharacters

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Representations of finite groups over $\mathbb{C}$

Theorem

Let Γ be a finite group and let $\rho_1 : \Gamma \to \mathrm{GL}_{n_1}(\mathbb{C})$, $\rho_2 : \Gamma \to \mathrm{GL}_{n_2}(\mathbb{C})$ ($n_1, n_2 \in \mathbb{N}_0$) be representations of Γ. Then

$$\forall \gamma \in \Gamma : \operatorname{tr} \rho_1(\gamma) = \operatorname{tr} \rho_2(\gamma) \quad \implies \quad \rho_1 \cong \rho_2$$

The map

$$\mathrm{Rep}(\Gamma) \to \mathbb{C}[\Gamma]^{\Gamma}, \quad \rho \mapsto \operatorname{tr} \rho$$

where $\mathrm{Rep}(\Gamma)$ is the set of conjugacy classes of representations is injective.

What is the image?

Every representation has a decomposition into irreducible representations, which is unique up to isomorphism.

$$\operatorname{tr} : \mathbb{N}_0[\mathrm{IrrRep}(\Gamma)] = \mathrm{Rep}(\Gamma)$$

$\operatorname{tr}(\mathrm{IrrRep}(\Gamma))$ is an orthonormal basis of $\mathbb{C}[\Gamma]^{\Gamma}$ with respect to the bilinear form

$$\langle \chi_1, \chi_2 \rangle := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_1(\gamma^{-1}) \chi_2(\gamma)$$

Representations of finite groups over $\mathbb{C}$

We would like to recover ρ from its character $\chi := \operatorname{tr} \rho$.

If $\{\rho_1, \dots, \rho_r\}$ are representatives of the irreducible representations of Γ, then

$$\chi = \sum_{i=1}^{r} \langle \chi, \operatorname{tr} \rho_i \rangle \operatorname{tr} \rho_i$$

for all $\chi \in \mathbb{C}[\Gamma]^\Gamma$.

In particular

$$V = \bigoplus_{i=1}^{r} \rho_i^{\oplus \langle \chi, \operatorname{tr} \rho_i \rangle}$$

for any representation ρ with $\chi = \operatorname{tr} \rho$.

In particular $\chi \in \mathbb{C}[\Gamma]^\Gamma$ is in the image of $\operatorname{tr} : \operatorname{Rep}(\Gamma) \to \mathbb{C}[\Gamma]^\Gamma$ if and only if $\langle \chi, \operatorname{tr} \rho_i \rangle \in \mathbb{N}_0$ for all i.

This is not a satisfactory answer, as we don't know the ρ_i a priori.

Representations of finite groups over $\mathbb{C}$

Is there a way to axiomatize characters of representations?

Suppose $\chi = \operatorname{tr}\rho$. We collect basic properies of χ.

- $\chi(1) = \dim\rho =: n$.
- $\chi(\gamma\gamma') = \chi(\gamma'\gamma)$ for all $\gamma, \gamma' \in \Gamma$.
-
$$\sum_{\sigma\in S_{n+1}} \operatorname{sign}(\sigma)\chi_\sigma(\gamma_1,\ldots,\gamma_{n+1}) = 0$$
  for all $\gamma_1,\ldots,\gamma_{n+1} \in \Gamma$. Here
$$\chi_\sigma(\gamma_1,\ldots,\gamma_{n+1}) := \prod_{c \text{ cycle of } \sigma} \chi_c(\gamma_1,\ldots,\gamma_{n+1})$$
  and for $c = (a_1 \ldots a_l)$, we put
$$\chi_c(\gamma_1,\ldots,\gamma_{n+1}) := \chi(\gamma_{a_1}\ldots\gamma_{a_l})$$
  This is a relation, that holds for any $(n+1)$-tuple of $n \times n$ matrices. Example for $n = 2$:
$$\chi(abc) + \chi(acb) - \chi(ab)\chi(c) - \chi(ac)\chi(b) - \chi(bc)\chi(a) + \chi(a)\chi(b)\chi(c) = 0$$

Pseudocharacters for $\mathrm{GL}_n(\mathbb{C})$

Definition (Taylor, 1991)

A d-dimensional *pseudocharacter* (for $\mathrm{GL}_n(\mathbb{C})$) is a map $T : \Gamma \to \mathbb{C}$, which satisfies the equations listed on the previous slide. [Tay, §1.1]

Any nontrivial examples?

Are there pseudocharacters, which do not come from representations?

The answer for finite groups over $\mathbb{C}$ is: No!

Theorem (Taylor, 1991)

If K is an algebraically closed field of characteristic 0 and $T : \Gamma \to K$ is a pseudocharacter, then there is a semisimple representation ρ of Γ with $T = \operatorname{tr} \rho$ and ρ is unique up to isomorphism. [Tay, Thm. 1]

What if K has positive characteristic?

We have to take $d \geq 2$ to find counterexamples, since for $d = 1$ the pseudocharacter relation

$$T(a)T(b) - T(ab) = 0$$

implies, that T is a homomorphism and therefore we just have $T = \operatorname{tr} T$.

A counterexample

Example

In characteristic 2 and dimension 2, uniqueness of the representation fails:

$$\rho : C_3 \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$$

with

$$\rho(\gamma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

where $\omega^2 + \omega + 1 = 0$ is semisimple and has the same pseudocharacter as the trivial representation, which is 0.

This ambiguity can be resolved by also keeping the determinant.

If we keep pairs $(\mathrm{tr}, \det)$, then what relations do we need?

Motivation for Lafforgue's definition

In general, we want to keep track of all coefficients of the characteristic polynomial of the matrices $\rho(\gamma)$ for all $\gamma \in \Gamma$.

Fact from classical invariant theory: $K[\mathrm{GL}_n]^{\mathrm{GL}_n}$ is generated by $s_1, \ldots, s_n$, where

$$\det(X - t) = \sum_{i=0}^{n} (-1)^i s_i(X) t^{n-i}$$

Any representation $\rho : \Gamma \to \mathrm{GL}_n(K)$ defines a homomorphism

$$\Xi_\rho : K[\mathrm{GL}_n]^{\mathrm{GL}_n} \to K[\Gamma]^\Gamma, \quad f \mapsto (\gamma \mapsto f(\rho(\gamma)))$$

$\Xi_\rho(\mathrm{tr})$ is Taylor's pseudocharacter of ρ and $\Xi_\rho(\det)$ is $\det(\rho)$.

If a general $\Xi : K[\mathrm{GL}_n]^{\mathrm{GL}_n} \to K[\Gamma]^\Gamma$ is given, can we recover a representation?

No: By Chevalley's restriction theorem $K[\mathrm{GL}_n]^{\mathrm{GL}_n} = K[T]^W$ is a polynomial ring in elementary symmetric functions.

Such a Ξ may assign any class function to each s_i. This is not what we want.

Motivation for Lafforgue's definition

The pseudocharacter relation in Taylor's definition depends on $n+1$ elements $\gamma_1, \dots, \gamma_{n+1}$ of Γ.

$$\sum_{\sigma \in S_{n+1}} \operatorname{sign}(\sigma) T_\sigma(\gamma_1, \dots, \gamma_{n+1}) = 0$$

Here T_σ can be seen as the image of an invariant function $\operatorname{tr}_\sigma \in K[\mathrm{GL}_n^{n+1}]^{\mathrm{GL}_n}$.

Example: If $\sigma = (12)(3) \in S_3$, then $\operatorname{tr}_\sigma = \operatorname{tr}(AB)\operatorname{tr}(C)$, where (A, B, C) are the generic matrices in GL_2^3.

The function tr_σ can be generated from tr by substitutions and multiplication of group elements.

The pseudocharacter relation becomes the relation

$$\sum_{\sigma \in S_{n+1}} \operatorname{sign}(\sigma) \operatorname{tr}_\sigma = 0$$

in $K[\mathrm{GL}_n^{n+1}]^{\mathrm{GL}_n}$.

Definition of G-pseudocharacters

Definition (G-pseudocharacter)

Let Γ be an abstract group, let G be a connected reductive group over $\mathcal{O}$ and let A be a commutative $\mathcal{O}$-algebra.A *G-pseudocharacter* Θ of Γ over A is a sequence of $\mathcal{O}$-algebra maps

$$\Theta_n : \mathcal{O}[G^n]^G \to \mathrm{Map}(\Gamma^n, A)$$

for each $n \geq 1$, satisfying the following conditions[1]:

1. For all $n, m \geq 1$, each map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, every $f \in \mathcal{O}[G^m]^G$ and all $\gamma_1, \dots, \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$$

   where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

2. For all $m \geq 1$, for all $\gamma_1, \dots, \gamma_{m+1} \in \Gamma$ and every $f \in \mathcal{O}[G^m]^G$, we have

$$\Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_{m+1}) = \Theta_m(f)(\gamma_1, \dots, \gamma_m\gamma_{m+1})$$

   where $\hat{f}(g_1, \dots, g_{m+1}) = f(g_1, \dots, g_m g_{m+1})$. [Laf, Def.-Prop. 11.3]

[1] Here G acts on G^n by $g \cdot (g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$. This induces a rational action of G on the affine coordinate ring $\mathcal{O}[G^n]$ of G^n. The submodule $\mathcal{O}[G^n]^G \subseteq \mathcal{O}[G^n]$ is defined as the rational invariant module of the G-representation $\mathcal{O}[G^n]$. It is an $\mathcal{O}$-subalgebra, since G acts by $\mathcal{O}$-linear automorphisms.

G-pseudocharacters

Theorem (V. Lafforgue, 2012)

Let Γ be a group, let K be an algebraically closed field, let G be a connected reductive group over K and let Θ be a pseudocharacter of Γ with values in G and coefficient ring K. Then there is a G-semisimple representation $\rho : \Gamma \to G(K)$, which is unique up to $G(K)$-conjugacy, such that $\Theta(\rho) = \Theta$. [Laf, Prop. 11.7]

Theorem (C. Procesi, 1976)

If $G = \mathrm{GL}_n$ and K has characteristic 0, then $\Theta \mapsto \Theta_1(\mathrm{tr})$ induces a bijection between Lafforgue's GL_n-pseudocharacters and Taylor's pseudocharacters.

This follows from [Pro, Thm. 1.3] unravelling the definitions.

Theorem

The functor $\mathrm{PC}_G^\Gamma : \mathcal{O}\text{-Alg} \to \mathrm{Set}$, which maps A to the set of A-valued G-pseudocharacters of Γ is representable by an affine $\mathcal{O}$-scheme.

Theorem

If Γ is finitely generated, then PC_G^Γ is of finite type over $\mathcal{O}$.

An alternative: Determinant laws

G. Chenevier introduced a notion of 'determinant law' [Che] which leads to a notion of pseudocharacter for GL_n over arbitrary base rings with similar properties.

Theorem (K. Emerson, 2018)

Chenevier's notion of determinant law is equivalent to Lafforgue's notion of GL_n*-pseudocharacter (over any base ring). [Eme]*

The proof uses deep results on invariant theory of matrices, i.e. generators and relations of the algebras $\mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$.

Generators of $\mathbb{Z}[G^m]^G$ are known for some classical groups, relations are mostly lacking. There is some work of Lopatin on generators for orthogonal groups.

Moduli of representations

G-pseudocharacters are a means to study moduli of representations up to conjugacy.

The functor

$$\mathcal{O}\text{-Alg} \to \text{Set}, \quad A \mapsto \text{Hom}(\Gamma, G(A))/G(A)$$

is in general not representable.

We can replace it by the GIT quotient $\text{Hom}(\Gamma, G) /\!\!/ G$ and there will be a comparison map

$$\nu : \text{Hom}(\Gamma, G) /\!\!/ G \to \text{PC}_G^\Gamma$$

Theorem (C. Wang-Erickson, 2015)

For $G = \text{GL}_n$, ν is an adequate homeomorphism[2].[WE, Thm. 2.20]

Theorem (Moakher, Q., 2022)

For symplectic determinant laws, ν is an integral universal homeomorphism. [this is an active project]

[2] integral universal homeomorphism + local isomorphism in characteristic 0

Deformations of Galois representations

Let Γ be a profinite group.

Given a continuous representation $\overline{\rho} : \Gamma \to G(W(\mathbb{F}))$ over a finite field $\mathbb{F}$, we can ask for lifts

$$\begin{array}{ccc} & & G(A) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \Gamma & \xrightarrow{\overline{\rho}} & G(\mathbb{F}) \end{array}$$

where A is a local artinian $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$.

The *deformation functor* $A \mapsto \mathrm{Lift}_{\overline{\rho}}(A)$ will always be representable by a complete noetherian local $W(\mathbb{F})$-algebra.

We can use *deformation spaces* to study representations $\Gamma \to G(\overline{\mathbb{Q}}_p)$ in families.

Deformations of G-pseudocharacters

We say a G-pseudocharacter $\Theta = (\Theta_m)_{m \geq 1}$ is *continuous*, if for all $m \geq 1$ the map $\Theta_m : \mathcal{O}[G^m]^G \to \mathrm{Map}(\Gamma^m, A)$ has image in the set of continuous maps $\mathcal{C}(\Gamma^m, A)$.

We write $\mathrm{cPC}_G^\Gamma(A)$ for the set of continuous G-pseudocharacters over A.

A lift of a continuous G-pseudocharacter $\overline{\Theta} \in \mathrm{cPC}_G^\Gamma(\mathbb{F})$ is a continuous G-pseudocharacter Θ over an artinian local $W(\mathbb{F})$-algebra A with $\Theta \otimes_{W(\mathbb{F})} \mathbb{F} = \overline{\Theta}$.

The pseudodeformation functor is representable by a complete local $W(\mathbb{F})$-algebra $R_{\overline{\Theta}}^{\mathrm{ps}}$.

Theorem

If Γ *satisfies Mazur's condition* Φ_p *and* G *is either* GL_n, SL_n, Sp_{2n}, GSp_{2n}, O_n *or* GO_n, *then* $R_{\overline{\Theta}}^{\mathrm{ps}}$ *is noetherian.*

There is a map $\mathrm{Spec}(R_{\overline{\rho}}^{\square}) \to \mathrm{Spec}(R_{\overline{\Theta}}^{\mathrm{ps}})$ and a study of the fibers provides insight about the structure of the framed deformation space.

Further results

Now $\Gamma = \Gamma_F$ is the absolute Galois group of a p-adic local field.

Theorem (Böckle, Juschka, 2019)

For $G = \mathrm{GL}_n$, the special fiber of the pseudodeformation space $\mathrm{Spec}(R^{\mathrm{ps}}_{\overline{\Theta}})$ *is equidimensional of dimension* $[F : \mathbb{Q}_p]n^2 + 1$*. [BJ, Thm. 1]*

Theorem (Böckle, Iyengar, Paškūnas, 2021)

1. *For $G = \mathrm{GL}_n$, $R^{\square}_{\overline{\rho}}$ is a local complete intersection, flat over $\mathcal{O}$ and of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$.[BIP, Thm. 1.1] In particular every continuous representation $\overline{\rho} : \Gamma_F \to \mathrm{GL}_n(k)$ has a lift to characteristic* 0*.*
2. *Suppose $p \nmid 2d$. Then crystalline lifts with regular Hodge-Tate weights are Zarski-dense in* $\mathrm{Spec}(R^{\square}_{\overline{\rho}}[1/p])$*.[BIP, Thm. 1.5]*

The Zariski-density of crystalline points played a role in the proof of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. [CDP]

We expect, that these results will be important for future developments in this direction.

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