Derived schemes

Seminar on derived algebraic geometry
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The goal of this talk is to define derived schemes.

1 $(\infty, 1)$-sheaves

We start by defining sheaves on topological spaces.

Definition 1.1. Let X be a topological space and let $\mathrm{Op}(X)$ be the category of open subsets of X. Let $\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Ani}$ be a presheaf. We say that $\mathcal{F}$ is a *sheaf*, if for any collection $\{U_i\}_i$ of open subsets with $U = \bigcup_i U_i$ and $\mathcal{U} = \{V \in \mathrm{Op}(X) \mid \exists i : V \subseteq U_i\}$ the map $\mathcal{F}(U) \to \lim_{V \in \mathcal{U}} \mathcal{F}(V)$ is an equivalence.

The above definition extends to sheaves with values in an arbitrary $(\infty, 1)$-category, [Lur18, Definition 1.1.2.1]. If the target category has a forgetful functor, which preserves and reflects limits, like the category of animated commutative rings, then the sheaf condition is the same for the underlying anima.

If $f : X \to Y$ is a continuous map between topological spaces and $\mathcal{F}$ is a sheaf on X, then we can define the pushforward $f_*\mathcal{F}$ by precomposition with the induced functor $f^{-1} : \mathrm{Op}(Y)^{\mathrm{op}} \to \mathrm{Op}(X)^{\mathrm{op}}$. The pushforward is indeed a sheaf, since f^{-1} is right cofinal.

2 Derived ringed spaces

We write $\mathrm{Shv}_{\mathrm{Ani(CRing)}}(X)$ for the category of sheaves of animated commutative rings on X. The association $X \mapsto \mathrm{Shv}_{\mathrm{Ani(CRing)}}(X)$ together with pushforward defines a functor $\mathrm{Shv}_{\mathrm{Ani(CRing)}} : \mathrm{Top} \to \mathrm{Cat}_\infty$. By applying the relative nerve construction [Lur09, Definition 3.2.5.2], we obtain a cocartesian fibration $\mathrm{dRS} \to \mathrm{Top}$ whose fibers are the categories $\mathrm{Shv}_{\mathrm{Ani(CRing)}}(X)$. The total category dRS is the category of pairs $(X, \mathcal{O}_X)$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of animated commutative rings on X. A morphism is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of animated rings. We refer to the objects as *derived ringed spaces*.

We write $\pi_n\mathcal{O}_X$ for the sheafification[1] of $U \mapsto \pi_n(\mathcal{O}_X(U))$. If $(X, \mathcal{O}_X)$ is a derived ringed space, then $(X, \pi_0\mathcal{O}_X)$ is a ringed space. We denote by $\mathrm{dRS}^{\leq n} \subseteq \mathrm{dRS}$ the full subcategory on n-truncated derived ringed spaces.

Proposition 2.1. The inclusion functor has a right adjoint $\tau_{\leq n} : \mathrm{dRS} \to \mathrm{dRS}^{\leq n}$, given on objects by $\tau_{\leq n}(X, \mathcal{O}_X) := (X, \tau_{\leq n}\mathcal{O}_X)$.

Definition 2.2. A *derived locally ringed space* is a derived ringed space $(X, \mathcal{O}_X)$, such that $(X, \pi_0\mathcal{O}_X)$ is locally ringed. The category dLRS of derived locally ringed spaces is defined as the fiber product of dRS with the category of locally ringed spaces over the category of ringed spaces.

3 The derived spectrum of an animated ring

Let A be an animated commutative ring. We define the underlying topological space $|\mathrm{Spec}(A)|$ as the underlying topologial space $|\mathrm{Spec}(\pi_0 A)|$.

[1] Sheafification is necessary, because truncation doesn't commute with arbitrary limits.

To define the structure sheaf on $X := |\operatorname{Spec}(A)|$, we proceed in two steps: First we define a sheaf on the category $\mathcal{B} \subseteq \operatorname{Op}(X)$ of basic open sets $D(f) = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$. Then we extend the sheaf to all of $\operatorname{Op}(X)$.

Proposition 3.1. Let X be a topological space and let $\mathcal{B}$ be a collection of open subsets of X satisfying the following conditions:

1. $\mathcal{B}$ forms a basis for the topology of X.
2. If $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$.
3. Each $U \in \mathcal{B}$ is quasi-compact.

Suppose $\mathcal{F} : \mathcal{B}^{\mathsf{op}} \to \mathrm{Ani}$ is a presheaf, satisfying the following descent condition:

> Let $U_1, \ldots, U_n \in \mathcal{B}$ be a finite collection of open sets with $U = \bigcup_i U_i \in \mathcal{B}$. For each subset $S \subseteq \{1, \ldots, n\}$, define $U_S := \bigcap_{i \in S} U_i$. Then $\mathcal{F}(U) \to \lim_{\emptyset \neq S} \mathcal{F}(U_S)$ is an equivalence, where the limit is taken over the category of all non-empty subsets of $\{1, \ldots, n\}$.

Then the right Kan extension of $\mathcal{F}$ to $\operatorname{Op}(X)^{\mathsf{op}}$ is a sheaf.

The proof is taken from [Lur18, Proposition 1.1.4.4].

Proof. Let $\{U_i\}_i$ be any collection of open subsets, let $U = \bigcup_i U_i$ and $\mathcal{U} = \{V \in \operatorname{Op}(X) \mid \exists i : V \subseteq U_i\}$. We need to show that $\mathcal{F}(U) \to \lim_{V \in \mathcal{U}} \mathcal{F}(V)$ is an equivalence.

Since $\mathcal{B}$ is a basis, the inclusion $\mathcal{B} \cap \mathcal{U} \to \mathcal{U}$ is right cofinal. Therefore $\lim_{V \in \mathcal{U}} \mathcal{F}(V) \simeq \lim_{V \in \mathcal{B} \cap \mathcal{U}} \mathcal{F}(V)$. Let us write $\mathcal{U}_U := \{V \in \operatorname{Op}(X) \mid V \subseteq U\}$. Again, since $\mathcal{B}$ is a basis, the inclusion $\mathcal{B} \cap \mathcal{U}_U \to \mathcal{U}_U$ is right cofinal, hence $\mathcal{F}(U) \simeq \lim_{V \in \mathcal{B} \cap \mathcal{U}_U} \mathcal{F}(V)$. It therefore suffices to show that $\mathcal{F}|_{\mathcal{B} \cap \mathcal{U}_U}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{B} \cap \mathcal{U}}$, i.e. that for every $V \in \mathcal{B} \cap \mathcal{U}_U$, the map

$$\theta : \mathcal{F}(V) \to \lim_{W \in \mathcal{B} \cap \mathcal{U}, W \subseteq V} \mathcal{F}(W)$$

is an equivalence. Since V is quasi-compact, we can write $V = V_1 \cup \cdots \cup V_n \in \mathcal{B} \cap \mathcal{U}$. Let $\mathcal{V} \subseteq \mathcal{B}$ be the collection of all $W \in \mathcal{B}$, that belong to (at least) one of the V_i.

We claim that $\mathcal{F}|_{\mathcal{B} \cap \mathcal{U}_V}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{V}}$. Assuming the claim, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\qquad\theta\qquad} & \lim_{W \in \mathcal{B} \cap \mathcal{U}, W \subseteq V} \mathcal{F}(W) \\ & \searrow \qquad \swarrow & \\ & \lim_{W \in \mathcal{V}} \mathcal{F}(W) & \end{array}$$

The left vertical map is an equivalence, since the inclusion $\{V_S \mid \emptyset \neq S \subseteq \{1, \ldots, n\}\} \to \mathcal{V}$ (definition of V_S similarly as for U_S) is right cofinal and the right vertical map is an equivalence by the claim.

To prove the claim, we must show that for every $W \in \mathcal{B}$ with $W \subseteq V$, the map

$$\phi : \mathcal{F}(W) \to \lim_{W' \subseteq W, W' \in \mathcal{V}} \mathcal{F}(W')$$

is an equivalence. Again, the inclusion $\{W_S \mid \emptyset \neq S \subseteq \{1, \ldots, n\}\} \to \mathcal{U}_W \cap \mathcal{V}$ is right cofinal, so the target of ϕ can be replaced by $\lim_{\emptyset \neq S} \mathcal{F}(W_S)$. Now descent follows from our assumption. $\square$

Before we can construct the structure sheaf, we need to explain that the étale site is invariant under truncation.

Proposition 3.2. Let A be an animated ring. The functor from étale A-algebras to étale $\pi_0(A)$-algebras induced by π_0 is an equivalence of ∞-categories.

Proof. This is proved in [CS23, Proposition 5.2.4]. □

We define $\mathcal{O}_X(D(f)) := A[f^{-1}] = A \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}]$, where $\mathbb{Z}[x] \to A, x \mapsto f$. That this defines a presheaf of A-algebras follows from the previous proposition and we also know that $A[f^{-1}]$ maps to $\pi_0(A[f^{-1}]) = \pi_0(A)[f^{-1}]$.

Proposition 3.3. Let A be an animated ring. Then there exists a sheaf of animated rings $\mathcal{O}$ on the topological space $X := |\operatorname{Spec}(\pi_0 A)|$ and a map of animated rings $A \to \mathcal{O}(X)$ with the following properties:

1. For every $f \in \pi_0 A$ defining a standard open subset $D(f) := \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$ the composite map $A \to \mathcal{O}(X) \to \mathcal{O}(D(f))$ induces an isomorphism $A[f^{-1}] \to \mathcal{O}(D(f))$.
2. The canonical map $\pi_0 A \to \pi_0(\mathcal{O}(X)) \to (\pi_0\mathcal{O})(X)$ induces an isomorphism $\mathcal{O}_{\operatorname{Spec}(\pi_0 A)} \to \pi_0\mathcal{O}$.

The proof is an adaption of the proof of [Lur18, Proposition 1.1.4.3] to the animated setting.

Proof. Let $\mathcal{B}$ be the collection of basic open subsets of X and let $\overline{\mathcal{O}}$ be the structure sheaf of $\operatorname{Spec}(\pi_0 A)$. By Proposition 3.2 we have a presheaf $\mathcal{O} : \mathcal{B}^{\mathsf{op}} \to \operatorname{Ani}(\operatorname{CRing})_{A/}$, and we know that $\mathcal{O}(D(f)) = A[f^{-1}]$. We define $\mathcal{O} : \operatorname{Op}(X)^{\mathsf{op}} \to \operatorname{Ani}(\operatorname{CRing})_{A/}$ as a right Kan extension.

We claim, that $\mathcal{O}$ is a sheaf of animated rings. We will verify the descent condition of Proposition 3.1.

Let $U_1, \dots, U_n$ be a collection of basic open subsets of X whose union $U = \bigcup_{i=1}^n U_i$ is also basic. For a subset $S \subseteq \{1, \dots, n\}$ write $U_S := \bigcap_{i \in S} U_i$. We need to show, that the canonical map

$$\mu : \mathcal{O}(U) \to \lim_{S \neq \emptyset} \mathcal{O}(U_S)$$

is an isomorphism, where the limit is taken over the partially ordered set of nonempty subsets of $\{1, \dots, n\}$. Since the U_i cover U the map $\overline{\mathcal{O}}(U) \to \prod_{i=1}^n \overline{\mathcal{O}}(U_i)$ is faithfully flat. By applying π_0 (using that π_0 preserves finite products) we see that the map $\mathcal{O}(U) \to \prod_{i=1}^n \mathcal{O}(U_i)$ is also faithfully flat. Since tensoring with a faithfully flat algebra is conservative, after tensoring with $- \otimes_{\mathcal{O}(U)} \prod_{i=1}^n \mathcal{O}(U_i)$ it is sufficient to see, that each

$$\mu_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \otimes_{\mathcal{O}(U)} \lim_{S \neq \emptyset} \mathcal{O}(U_S)$$

is an isomorphism. Since $\mathcal{O}(U_i)$ is flat over $\mathcal{O}(U)$, the functor $- \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) : D(\mathcal{O}(U)) \to D(\mathcal{O}(U_i))$ is an exact functor between stable ∞-categories and therefore preserves finite limits. So μ_i can be identified with the canonical map

$$\mathcal{O}(U_i) \to \lim_{S \neq \emptyset} (\mathcal{O}(U_i) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_S)) \simeq \lim_{S \neq \emptyset} \mathcal{O}(U_{S \cup \{i\}})$$

The poset over which the limit $\lim_{S \neq \emptyset} \mathcal{O}(U_{S \cup \{i\}})$ is taken contains an initial object, so this limit coincides with $\mathcal{O}(U_i)$. This completes the proof of (1).

For (2), we see that $U \mapsto \pi_0(\mathcal{O}(U))$ is a presheaf, which coincides with $\overline{\mathcal{O}}$ on $\mathcal{B}$ by construction. Since this is already a sheaf on $\mathcal{B}$, we have $(\pi_0\mathcal{O})(U) = \pi_0(\mathcal{O}(U))$ for all $U \in \mathcal{B}$. As $\mathcal{B}$ is a basis for the topology on X, we can apply [Sta19, 009N] to deduce that $\pi_0\mathcal{O}$ and $\overline{\mathcal{O}}$ coincide. □

4 Derived schemes

Definition 4.1. Let A be an animated commutative ring. We define the *structure sheaf* on $\mathrm{Spec}(A) := \mathrm{Spec}(\pi_0 A)$ as the sheaf constructed in Proposition 3.3. We write $\mathrm{Spec}(A)$ for the resulting derived locally ringed space.

Definition 4.2. A derived locally ringed space $(X, \mathcal{O}_X)$ is a *derived scheme*, if X has an open covering $X = \bigcup_i U_i$, such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $\mathrm{Spec}(A)$ for some animated commutative ring A. The category dSch of derived schemes is defined as a full subcategory of dLRS.

Proposition 4.3. Let $(X, \mathcal{O}_X)$ be a derived locally ringed space and let A be an animated commutative ring. Then composition with the canonical map $A \to \mathcal{O}(\mathrm{Spec}(A))$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{dLRS}}((X, \mathcal{O}_X), \mathrm{Spec}\, A) \to \mathrm{Hom}_{\mathrm{Ani(CRing)}}(A, \mathcal{O}_X(X))$$

The proof is an adaption of the proof of [Lur18, Proposition 1.1.5.5] to the animated setting.

Proof. Let $\phi : A \to \mathcal{O}_X(X)$ be a map of animated rings. We want to show that the fiber

$$Z := \mathrm{Hom}_{\mathrm{dLRS}}((X, \mathcal{O}_X), \mathrm{Spec}\, A) \times_{\mathrm{Hom}_{\mathrm{Ani(CRing)}}(A, \mathcal{O}_X(X))} \{\phi\}$$

is contractible.

Let $R := \pi_0 A$.

For $x \in X$ the map $A \to \mathcal{O}_X(X) \to (\pi_0 \mathcal{O}_X)_x$ induces a map $R \to (\pi_0 \mathcal{O}_X)_x \to \kappa(x)$. The kernel is a prime ideal $\mathfrak{p}_x \subseteq R$. This defines a map of sets

$$g : X \to |\,\mathrm{Spec}(A)|,\ x \mapsto \mathfrak{p}_x.$$

One can easily check that g is continuous, we refer to [Lur18, Proposition 1.1.5.5] for the details.

Moreover, Z can be identified with the fiber (we omit some details here)

$$\mathrm{Hom}_{\mathrm{Shv}_{\mathrm{Ani(CRing)}}(|\,\mathrm{Spec}(A)|)}(\mathcal{O}, g_* \mathcal{O}_X) \times_{\mathrm{Hom}_{\mathrm{Ani(CRing)}}(A, \mathcal{O}_X(X))} \{\phi\}.$$

Here $\mathcal{O}$ is the structure sheaf of $\mathrm{Spec}\, A$.

For the collection $\mathcal{B}$ of basic open subsets of $\mathrm{Spec}\, A$ we have a fully faithful functor

$$T : \mathrm{Shv}_{\mathrm{Ani(CRing)}}(\mathrm{Spec}\, A) \to \mathrm{Fun}(\mathcal{B}^{\mathsf{op}}, \mathrm{Ani(CRing)}),$$

by Proposition 3.1. We can therefore identify Z with

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{B}^{\mathsf{op}}, \mathrm{Ani(CRing)})}(T\mathcal{O}, T g_* \mathcal{O}_X) \times_{\mathrm{Hom}_{\mathrm{Ani(CRing)}}(A, \mathcal{O}_X(X))} \{\phi\}$$

Since $T\mathcal{O}$ is pointwise a localization of A, which is étale over A, we can use Proposition 3.2 to identify Z with

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{B}^{\mathsf{op}}, \mathrm{CRing}_R)}(\pi_0 T\mathcal{O}, \pi_0 T g_* \mathcal{O}_X)$$

This is already just a point or empty. For $r \in R$, it suffices to check that the image of r in $\pi_0(g_* \mathcal{O}_X)(D(r))$ is invertible.

Multiplication by r defines an endomorphism of $\mathcal{O}_X$. Let $U := g^{-1}(D(r))$. For all $x \in X$ the image of r in $(\pi_0 \mathcal{O}_X)_x$ is invertible. Since U is open, there is a neighborhood V of x in U, such that r is an invertible endomorphism of $\mathcal{O}_X|_V$. This means that the fiber of r as an endomorphism of $\mathcal{O}_X(U)$ is 0, so r is invertible in $\pi_0(\mathcal{O}_X(U)) = \pi_0(g_* \mathcal{O}_X)(D(r))$. $\square$

Proposition 4.3 shows that the association $A \mapsto \mathrm{Spec}(A)$ defines a fully faithful right adjoint $\mathrm{Spec} : \mathrm{Ani(CRing)}^{\mathsf{op}} \to \mathrm{dSch}$.

Proposition 4.4. The construction $(X, \mathcal{O}_X) \mapsto (X, \pi_0\mathcal{O}_X)$ induces an equivalence $\mathrm{dSch}^{\leq 0} \simeq \mathrm{Sch}$.

At the end, I want to remark that as in the 1-categorical situation, the Yoneda embedding

$$\mathrm{Ani}(\mathrm{CRing})^{\mathsf{op}} \to \mathrm{Fun}(\mathrm{Ani}(\mathrm{CRing}), \mathrm{Ani})$$

extends to a fully faithful functor

$$\begin{aligned} \mathrm{dSch} &\to \mathrm{Fun}(\mathrm{Ani}(\mathrm{CRing}), \mathrm{Ani}), \\ (X, \mathcal{O}_X) &\mapsto (A \mapsto \mathrm{Hom}_{\mathrm{dSch}}(\mathrm{Spec}\, A, (X, \mathcal{O}_X))) \end{aligned}$$

This is the analog of [Lur18, Proposition 1.6.4.2]. I will not have the time to explain this in detail in this talk, but we will return to the functorial perspective later in the seminar.

References

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